Isoperimetric problems in a sector

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Abstract. As we know the origin of isoperimetric problem is the problem confronted by Queen Dido. The problem was to find the shape of the boundary that should be laid down to enclose maximum area. If one assumes a straight coastline, the answer is semicircle. Some years ago, my colleage Ninh Van Thu told me Polya's question : what happens if the coastline is not straight, the region lies in a sector, and the two ends of the boundary lie on the two sides of the sector? In 1983, P.-L. Lions and F. Pacella gave the answer for this question in a convex cone in $\mathbb{R}^n, n \geq 2$. Recently X. Cabre et al. gave another proof of this result. In this note I give an elementary proof of this result for $n = 2$.

Some continuous isoperimetric inequalities

Theorem

When $0 < \alpha < \pi$ *the isoperimetric inequality*

$$A \leq \frac{L^2}{2\alpha}$$

holds. The equality happens iff C *is a part of a circle with center at the vertex* O.

Theorem

When $\pi < \alpha \leq 2\pi$ *the isoperimetric inequality*

$$A \leq \frac{L^2}{2\pi}$$

holds. The equality happens iff C *is a semicircle which has an end at the vertex* O.

Some discrete isoperimetric inequalities

Theorem

When $0 < \alpha < \pi$ *the isoperimetric inequality*

$$A_n \leq \frac{L_n^2}{4n\tan(\frac{\alpha}{2n})}$$

holds. The equality happens iff

$$|OP_0| = |OP_1| = \cdots = |OP_n|, \angle P_j OP_{j+1} = \frac{\alpha}{n}, j = 0, 1, \ldots, n-1.$$

When $\pi < \alpha \leq 2\pi$ *the isoperimetric inequality*

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holds. The equality happens iff

$$O \equiv P_n, |P_j P_{j+1}| = \frac{L_n}{n}, \ \angle P_j OP_{j+1} = \frac{\pi}{2n}, \ j = 0, 1, \ldots, n-1.$$