STRUCTURE OF SEMIRINGS

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1. Abstract

Additive and multiplicative structures play an important role in determining the structure of semirings. A semiring is said to be a Positive Rational Domain (PRD) if (S,•)is an abelian group. In [2] Satyanarayana proved that if a totally ordered semiring (S,+,•) contains the multiplicative identity, then (S, +) is non-negatively ordered or non-positively ordered. In this paper, we study the structure of semirings which are Positive Rational Domains (PRDs). We prove that the semiring of non-negatively ordered elements is isomorphic to the semiring of non-positively ordered elements. We study the conditions under which (S, +) is positively ordered or negatively ordered in PRD semirings. We also study the properties of semirings in which (S, +) is a zeroid.

2. Preliminaries:

A triple (S, +, •) is called a semiring if (S, +) is a semigroup; (S, •) is a semigroup; a(b + c) = ab + ac and (b + c)a = ba + ca for every a, b, c in S. A semiring (S, +, •) is said to be totally ordered if there exists a full ordering on S under which (S, +) and (S, •) are totally ordered semigroups. A totally ordered (t.o) semigroup (S, •) is non–negatively (non – positively) ordered if $x^2 \geq x$ ($x^2 \leq x$) for every x in S. Zeroid of a semiring (S, +, •) is the set of all x in S such that x + y = y or y + x = y for some y in S.

A semiring is said to be monosemiring if ab = a + b for all a, b in S. A semiring is said to be subidempotent semiring if S satisfies the identity a + ab = b for every a, b in S. A semigroup (S, ·) is said to be left (right) singular if ab = a (ab = b). A semiring is said to be almost idempotent if a + a^2 = a^2 for all a in S. An element x is said to be additively regular if for each x in S there exists y in S such that x + y + x = x. A semiring S is multiplicatively subidempotent if and only if a + a^2 = a for all a in S. A semiring S is said to be an idempotent if a + a = a and a^2 = a for all a in S. A semiring is said to be a viterbi semiring in which S is additively idempotent and multiplicatively subidempotent.

A semiring S is said to be zero square if a^2 = 0 for all a in S. For an element a in a semiring is said to be complemented if there exists b in S such that if a + b = 1 and ab = ba = 0. A semiring is said to be periodic if $x^n = x^m$ for some natural numbers n and m where n and m depend on x.

Example 2.1:

If a t.o. semiring contains multiplicative identity, then (S, +) is non-negatively ordered or non-positively ordered. The converse is not necessarily true. This is evident from the following example. Here S = {1/4, 2/4, 3/4, 1}

+	1/4	2/4	3/4	1
1/4	3/4	3/4	1	1
2/4	3/4	1	1	1
3/4	1	1	1	1
1	1	1	1	1

.	1/4	2/4	3/4	1
1/4	2/4	3/4	1	1
2/4	3/4	1	1	1
3/4	1	1	1	1
1	1	1	1	1

Here (S, +) is non- negatively ordered and semiring S does not contain the multiplicative identity.

3. POSITIVE RATIONAL DOMAIN:

Definition 3.1 A semiring (S, +, •) is said to be a positive rational domain (PRD) if and only if (S, •) is an abelian group.

Definition 3.2: An element a of a semigroup (S, +) is completely regular if a + x + a = a and a + x = x + a and (S, +) is completely regular if every element of (S, +) is completely regular.

Definition 3.3: An element a of a semiring S is completely regular if there exists x ∈ S such that (A) a + x + a = a (B) a + x = x + a and (C) a(a + x) = a + x. A semiring S is said to be completely regular if every element a of S is completely regular.

Definition 3.4: A semiring S is said to be Boolean like semiring, if

(i) ab (a + b + ab) = ab, for all a, b in S and a.0 = 0.a = 0.

(ii) Weak commutative (abc = bac) for all a, b, c in S.

Theorem 3.5: Let S be a PRD and subidempotent semiring. Then the following are true:

(a) 1 + b = $a^{-1}b$ for all a, b in S.

(b) b^{-1} + $b^{-1}a$ + a = a^{-1}

Theorem 3.6: If S is a PRD and Z is a zeroid of S, then S = Z

Theorem 3.7: Let S be a PRD and a∈ C, Where C is the set of completely regular elements. Then C ={1}.

Theorem 3.8: Let (S, +, •) be a PRD semiring. Assume that |S| > 1. Then the following are true.

(1) S = $T_1 \cup T_2$, $T_1 \cap T_2$ = {e}, where (T_1,•) is a non-negatively ordered semigroup,(T_2,•) is a non-positively ordered semigroup and e is the multiplicative identity.

(2) (T_1, •) ≅(T_2, •)

Theorem 3.9: Let (S, +, •) be a PRD semiring satisfying the identity ab = a + b + ab for all a, b in S. If (S, +) is p.t.o (n.t.o), then 1 is maximum (minimum) element.

Theorem 3.10: Let S be a boolean like semiring and PRD. If (S, +) is p.t.o then 1 is the maximum element.

4.Some Special Classes of Semirings:

Theorem 4.1: Let (S, +, •) be a totally ordered semiring satisfying the identity ab = a + b + ab for all a, b in S. If (S, +) is p.t.o, then (S, •) is p.t.o.

Theorem 4.2: Let S be a PRD and a is an additively completely regular element and (S, +) is p.t.o. Then a is the maximum element

Theorem 4.3: Let |S|>1 and S be a PRD. If (S, +) is a completely regular semigroup. Then C is an infinite set where C be the set of all completely regular elements

Theorem 4.4: Let S be a complemented semiring and x be an additively completely regular element. Then the following are true.

(i) xy = yx = y.

(ii) y + y^2 = y^2 + y.

(iii) y is an additively completely regular element.

(iv) There are infinitely many completely regular elements.

Theorem 4.5: Let S be a zero square semiring in which x is an additively completely regular element. Then xy = 0 where y arises from the definition of completely regular.

Theorem 4.6: If S is a completely regular semiring and multiplicatively subidempotent, then (S, ·) is periodic.

Theorem 4.7: Let S be a completely regular semiring and a + ax = x. If (S, +) is left cancellative, then a^2 = 2a, for all a in S.

Theorem 4.8: Let S be a completely regular semiring in which a + ax = a, for a in S. If (S, +) is left cancellative, then a + x = (a + x)x.

Theorem 4.9: Let S be a completely regular and monosemiring. Then

(i) S is multiplicatively subidempotent.

(ii)a (a + ax) = a + x.

Theorem 4.10: Let S be a t.o. completely regular semiring and (S, +) be p.t.o. Then S is an idempotent semiring.

Example 4.11: we can illustrate the theorem with example in two different cases

(i) Here S = {a, x} and x < a

+	a	x
a	a	a
x	a	x

.	a	x
a	a	a
x	a	x

(ii)Here S = {0, a, x} and 0 < x < a

+	0	a	x
0	0	a	x
a	a	a	a
x	x	a	x

.	0	a	x
0	0	0	0
a	0	a	a
x	0	a	x

Theorem 4.12: Let S be a t.o.completely regular semiring and (S, ·) be p.t.o, Then the following are true

(i) (S,+)isnon-positively ordered.

(ii) If (S, +) is left cancellative, then (S, ·) is right singular.

(iii) If (S, +) is right singular, then (S, +) is p.t.o.

Theorem 4.13: Let S be an almost idempotent semiring.

(i) If S is viterbi semiring, then S is idempotent semiring.

(ii) If (S, +) is a zeroid and right cancellative, then (S, ·) is semilattice and regular.

Example 4.14:

(i)

+	a	x
a	a	a
x	a	x

.	a	x
a	a	a
x	a	x

(ii)

+	0	a	x
0	0	a	x
a	a	a	a
x	x	a	x

.	0	a	x
0	0	0	0
a	0	a	a
x	0	a	x

(iii)

+	0	x	b	y	a
0	0	x	b	y	a
x	x	x	a	a	a
b	b	a	b	b	a
y	y	a	b	y	a
a	a	a	a	a	a

.	0	x	b	y	a
0	0	0	0	0	0
x	0	x	a	a	a
b	0	a	b	b	a
y	0	a	b	y	a
a	0	a	a	a	a

A **monosemiring** is multiplicatively subidempotent semiring but a multiplicatively subidempotent semiring need not be a monosemiring. We can see this in example 1.

In Example 1

- S is an idempotent semiring
- S contains multiplicative identity which is also an additive identity
- S is a mono semiring
- S is multiplicatively subidempotent semiring
- (S, ·) is commutative and
- (S, ·) is not completely regular semigroup

In Example 2

- S is an idempotent semiring
- S contains multiplicative identity and additive identity
- S is multiplicatively subidempotent semiring
- (S, ·) is commutative and
- (S, ·) is not completely regular semigroup

In Example 3

- S is an idempotent semiring
- S contains multiplicative identity
- (S, ·) is commutative
- S is multiplicatively subidempotent semiring and
- (S, ·) is not completely regular semigroup

References:

[1] M. Satyanarayana, On the additive semigroup structure of Semirings. Semigroup Forum23(1981), 7-14.

[2]M.Satyanarayana, On the additive semigroup structure of ordered semirings. Semigroup Forum 31(1985), 193-199.

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